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INVESTIGATION OF THE FREE VIBRATIONS OF NONCIRCULAR CYLINDRICAL SHELLS

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An asymptotic method of integrating the free vibrations equations of a noncircular cylindrical shell, freely supported along the curvilinear edges, is proposed. The qualitative singularities of the vibrations associated with the fact that the shell is essentially noncircular are clarified. Numerical results are presented for the free vibrations frequencies and modes of a box shell which are compared to the results of the asymptotic analysis. It is supposed that the variability of the stress and strain states is great.

1. The bending equations of vibration of a cylindrical shell of arbitrary outline in displacements are, when second degree terms are discarded

$$\left[\frac{\partial^2}{\partial\alpha^2}+\frac{1-\sigma}{2}\frac{\partial^2}{\partial\beta^2}+(1-\sigma^2)\lambda\right]\xi+\frac{1+\sigma}{2}\frac{\partial^2\eta}{\partial\alpha\,\partial\beta}-\frac{\sigma}{R}\frac{\partial\zeta}{\partial\alpha}=0 \quad (1.1)$$

$$\frac{1+\sigma}{2}\frac{\partial^2\xi}{\partial\alpha\,\partial\beta}+\left[\frac{\partial^2}{\partial\beta^2}+\frac{1-\sigma}{2}\frac{\partial^2}{\partial\alpha^2}+(1-\sigma^2)\lambda\right]\eta-\frac{\partial}{\partial\beta}\left(\frac{\zeta}{R}\right)=0$$

$$\frac{\sigma}{R}\frac{\partial\xi}{\partial\alpha}+\frac{1}{R}\frac{\partial\eta}{\partial\beta}-\left[\frac{1}{R^2}-(1-\sigma^2)\lambda+\frac{h^2}{3}\left(\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}\right)^2\right]\zeta=0$$

$$\left(\lambda=\frac{m\omega^2r^2}{2Eh},\ \xi=2Ehu,\ \eta=2Ehv,\ \zeta=2Ehw\right)$$

The notation from the monograph [1] is used here and it is considered that α is the dimensionless length of the shell generator, β is the dimensionless length of the cross-sectional perimeter, $R=R(\beta)$ is the dimensionless radius of curvature of the cross section, h is half the dimensionless shell thickness (we consider $h\ll 1$). The dimensionless frequency parameter λ and the quantities ξ, η, ζ are defined by the formulas in the parentheses in (1.1) (for brevity the symbols ξ, η, ζ are called displacements), where m is the mass per unit area of the middle surface, ω is the vibration frequency, and r is some number characteristic for the shell with the dimensionality of a length. All the dimensionless quantities in the problem are referred to r. In writing (1.1) it is considered that the shell performs harmonic oscillations and the factor $\sin\omega t$ is omitted from the sought quantities.

Let us first consider a closed circular shell, supported freely at the circular edges. Let us seek the solution as

$$\xi=\xi_0\cos k\alpha\sin n\beta,\quad \eta=\eta_0\sin k\alpha\cos n\beta,\quad \zeta=\zeta_0\sin k\alpha\sin n\beta \quad (1.2)$$

Here n is an integer and ξ_0, η_0, ζ_0 are to be determined. Since n is an integer, then the condition that the shell be closed is automatically satisfied if r is considered the radius of its cross section for a circular shell.

Let us assume that there is one half-wave along the generator since the shell would be divided into several sections in the case of several half-waves, each of which would behave identically. Then the free support condition at the edges $\alpha = 0$ and $\alpha = l$ yields the equation $kl = \pi$ to determine k. In the following analysis we shall consider that l, and consequently k, are quantities of the order of unity; this also refers to noncircular shells.

Let us examine vibrations with sufficiently high variability, hence we assume

$$h^{-1} \gg n \gg 1 \tag{1.3}$$

The left inequality yields the domain of applicability of the initial equations (1.1).

Substituting (1.2) into (1.1), we obtain a homogeneous linear system of three equations with three unknowns for ξ_0, η_0, ζ_0. Equating its determinant to zero, we obtain a cubic equation for $\lambda_0 = (1-\sigma^2)\lambda$

$$\lambda_0^3 - \frac{3-\sigma}{2}n^2\lambda_0^2 + \frac{1-\sigma}{2}n^4\lambda_0 - \frac{(1-\sigma)^2(1+\sigma)}{2}k^4 - \frac{h^2}{6}(1-\sigma)n^8 = 0 \tag{1.4}$$

in which only the terms essential under the assumptions (1.3) have been kept. (Let us note that it is also possible to arrive at (1.4) by using formulas which are obtained from (1.2) by replacing $\sin n\beta$ therein by $\cos n\beta$ and $\cos n\beta$ by $\sin n\beta$.) By keeping in mind the assumptions relative to the orders of h, k, n, it can be shown directly that (1.4) will decompose into a quadratic and a linear equation to the accuracy of infinitesimals on the order of $O(\max\{n^{-6}; h^2n^2\})$, which means that we obtain for the roots of (1.4) to the same accuracy

$$\lambda_{01} \approx n^2, \quad \lambda_{02} \approx \frac{1-\sigma}{2}n^2, \quad \lambda_{03} \approx (1-\sigma^2)\frac{k^4}{n^4} + \frac{h^2}{3}n^4 \tag{1.5}$$

A graph of λ as a function of n for fixed h and k is pictured schematically in Fig. 1.

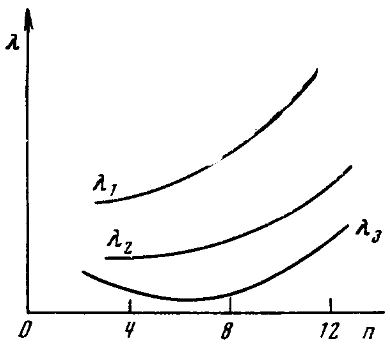

Fig. 1

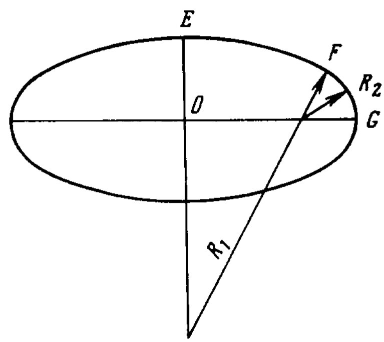

Fig. 2

By calculating the displacements corresponding to λ_1 and λ_2 it can be seen that the inequality $\zeta \ll \max(\xi, \eta)$ is valid (quasi-tangential vibrations [2]), but the inequality $\zeta \gg \max(\xi, \eta)$ is valid for displacements corresponding to λ_3 (quasi-transverse vibrations [2]). Let us distinguish quasi-transverse vibrations with medium ($1 \ll n \ll h^{-1/4}$), intermediate ($n \sim h^{-1/4}$) and high ($n \gg h^{-1/4}$) variabilities. For shells of nonpositive Gaussian curvature, the least vibrations frequency is achieved for quasi-transverse vibrations with intermediate variability [2]. Hence, we obtain from (1.5) for λ_{03} that the least frequency parameter λ is on the order of h for the circular cylindrical shells considered here.

2. Before considering cylindrical shells of arbitrary outline, let us present numerical results referring to finding the free vibrations frequencies and modes of box shells.

Box shells of symmetric oval cross sections (Fig. 2) were examined. Because of symmetry, let us consider the section EFG. Let the arcs EF and FG have the radii of curvature R_1 and R_2, respectively. If γ denotes the ratio between the semi-axes OG and OE of the oval, then by altering R_1 and R_2 and the angular measure of the arcs EF and FG in a suitable manner, smooth ovals with different γ and the very same perimeter can be obtained. It is assumed that the box shell is freely supported at the endfaces. Keeping this in mind, as well as the constancy of the coefficients of the system (1.1) in α we can seek the solution for the box shell as

$$\xi = \xi_* (\beta) \cos k\alpha,\ \eta = \eta_* (\beta) \sin k\alpha,\ \zeta = \zeta_* (\beta) \sin k\alpha\, (k = \pi/l \qquad (2.1)$$

The radius of curvature of the oval is a piecewise constant function of β, hence for each R_ν $(\nu = 1.2)$ the solution can be sought in the form

$$\{\xi_* \eta_*, \zeta_*\} = \exp (\theta_\nu \beta) \{\xi_0, \eta_0, \zeta_0\}.$$

The characteristic equations for θ_ν are of eighth order, and we hence obtain eight solutions for each R_ν. Let us limit ourselves to finding the symmetric vibrations modes. The frequency equation is then obtained from the condition of smoothness of the solution at the point of discontinuity F of the radius of curvature of the symmetric oval, where four solutions corresponding to the symmetric vibrations modes are taken on both EF and FG

A program to find the frequncies and their corresponding symmetric vibrations modes for box shells, whose cross sections are symmetric ovals, was compiled on the basis of the scheme described, and numerical results were obtained on the BESM-4 computer. The computations were carried out for certain values of h and γ, where $\sigma = 0.3$, $l = \pi$, $p = 4\pi/3$ in all cases and p is the dimensionless perimeter of the shell cross section (we assume $r = 1$).

Values of the least λ corresponding to the symmetric vibrations modes are presented below (see Table 1).

Table 1

h	γ		
	1.5	2	3
10^{-2}	—	$0.7876\cdot10^{-2}$	—
10^{-3}	$1.155\cdot10^{-3}$	$0.7532\cdot10^{-3}$	$0.3704\cdot10^{-3}$
10^{-4}	—	$0.7404\cdot10^{-4}$	—

It is seen that the λ_{min} decrease as the shells flatten out; this is explained by the diminution of the shell stiffness. It is also seen that for box shells whose vibrations modes are symmetric $\lambda_{min} \sim h$ as for circular shells. A computation showed that the vibrations corresponding to the λ_{min} presented are quasi-transverse.

The following 49 first values of λ were found for shells with $h = 10^{-3}$ and $\gamma \approx 2$ (see Table 2)

Table 2

0.000753	0.00137	0.00275	0.00367	0.00599	0.00660	0.0106
0.01154	0.0194	0.03191	0.04921	0.06853	0.07556	0.1049
0.1455	0.1964	0.2609	0.3388	0.4326	0.5467	0.6795
0.7000	0.8349	0.8951	1.019	1.228	1.468	1.746
2.047	2.114	2.409	2.804	3.243	3.730	4.282
4.653	4.904	5.555	6.280	6.636	7.084	7.956
8.904	9.938	11.06	12.27	13.20	13.59	15.00

Quasi-transverse vibrations correspond to the first 23 frequencies; for the next values of λ the quasi-transverse vibrations alternate with quasi-tangential and vibrations of general form (i.e. vibrations in which none of the displacement vector components predominates). It is seen that the density of the vibrations frequencies diminishes as the number of the frequency grows.

Let us present a graph for the predominant displacement ζ_* corresponding to the first, fourth, fifth and seventeenth frequencies (see Figs. 3 and 4, where the numbers on the curves correspond to the numbers of the frequencies; the curve *5* is given by dashes for clarity).

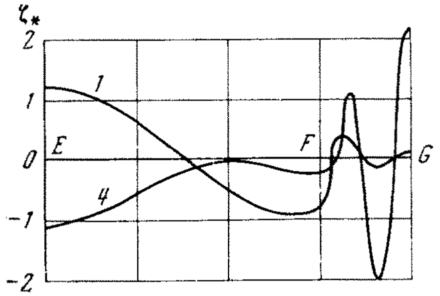

Fig. 3

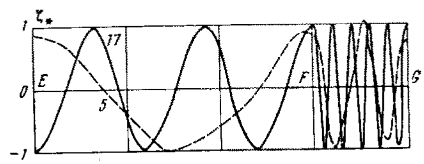

Fig. 4

Graphs are given for the section EFG of the shell since they can be continued by symmetry on the remaining portion, where the length scale for the portions EF and FG are the same. Let us note that the angular measure of the arc FG is twice the arc EF. It follows from the graph of ζ_* corresponding to the lowest frequency that the function ζ_* damps on FG by oscillating, hence there are two pairs of pure imaginary roots for θ_1 on the section EF and all eight roots are completely complex for θ_2 on FG. The vibrations modes and the roots θ_v for the second and third frequencies possess the same property although the damping of the function ζ_* on FG is less abrupt here. A graph of the ζ_* mode corresponding to the quarter frequency shows that the function ζ_* does not damp out on the section FG; the fact that there are two pairs of pure imaginary roots for θ_2 on FG corresponds to this (the remaining four roots are completely complex). The nature of the roots of the characteristic equation for the one-fifth frequency is the same as for the quarter frequency.

It can be concluded from the number of nodal lines that the variability of the mode ζ_* corresponding to λ_1 is greater than the mode ζ_* corresponding to λ_4 and less than the variability of ζ_* corresponding to λ_5. The mode ζ_* for λ_{17} possesses such variability that the variability of the radius of curvature is not essential (here there are two pairs of pure imaginary roots and four real for θ_1 and θ_2). Let us note that the mode ζ_* corresponding to λ_1 has a qualitative difference from the mode of circular shell vibrations corresponding to the lowest frequency since there is no zone where the vibrations decay for circular shells. The damping phenomenon detected here is related essentially to the variability of the radius of shell curvature, and as computations have shown, the damping will be the sharper the greater the γ differs from unity.

3. Now let us assume that $R^{-1}(\beta)$ is an arbitrary positive sufficiently smooth function. As befor, let us consider the shell simply supported along the curvilinear edges and hence the solution can be sought in the form (2.1) by assuming that k is found from the formula in (2.1). Let us examine the vibrations of open shells, i.e. vibrations with homogeneous boundary conditions on the rectilinear lines $\beta = 0$ and $\beta = \beta_0$.

Upon substitution of (2.1) into the system (1.1), we obtain a system of eighth order ordinary differential equations with coefficients dependent on β for ξ_*, η_*, ζ_*. Being interested in the state of stress with sufficiently high variability, let us seek the solution of this system by the method of exponential representation

$$\xi_* = \exp(f/\varepsilon)\ (\varepsilon^2\xi_1 + \varepsilon^3\xi_2 + \ldots), \qquad \eta_* = \exp(f/\varepsilon) \quad (\varepsilon\eta + \varepsilon^2\eta_2 + \ldots) \tag{3.1}$$
$$\zeta_* = \exp(f/\varepsilon)\ (\zeta_1 + \varepsilon\zeta_2 + \ldots), \quad \lambda = \varepsilon^4\lambda_1 + \varepsilon^6\lambda_2 + \varepsilon^7\lambda_3 + \ldots$$

Let us define the small parameter ε in (3.1) thus:

$$\varepsilon = (h / \sqrt{3(1-\sigma^2)})^{1/4} \tag{3.2}$$

The f, ξ_ν, η_ν, ζ_ν in (3.1) are the desired functions of β but λ_ν are the desired constants. The parameter ε defined by (3.2) and the powers of ε for ξ_1, η_1, ζ_1 in (3.1) are selected such that the correspondence of the desired solution with the quasi-transverse vibrations with intermediate variability is obtained in the sense that the modes and frequencies of the quasi-transverse vibrations with intermediate variability of circular shells possess the same asymptotic orders as the principal terms in ξ_*, η_*, ζ_* and λ of (3.1). However, such a correspondence does not mean that (3.1) hold everywhere, about which more below. The quantity f is called the variability function, and ξ_ν, η_ν, ζ_ν are intensity coefficients.

Substituting (3.1) into the above-mentioned system of ordinary differential equations and equating coefficients of identical powers of ε in each of the system equations, we obtain a set of recursion systems of linear algebraic equations for the intensity coefficients ξ_ν, η_ν, ζ_ν ($\nu = 1,2, \ldots$). The system for ξ_1, η_1, ζ_1 is homogeneous and in the following form (the primes denote the derivatives with respect to β):

$$\frac{1-\sigma}{2} f'^2\xi_1 + k\frac{1+\sigma}{2} f'\eta_1 - \frac{k\sigma}{R}\zeta_1 = 0 \tag{3.3}$$
$$f'^2\eta_1 - \frac{f'}{R}\zeta_1 = 0, \qquad \frac{f'}{R}\eta_1 - \frac{\zeta_1}{R^2} = 0$$

The determinant of the system (3.3) vanishes for any variability function f.

The algebraic systems for ξ_ν, η_ν, ζ_ν ($\nu > 1$) are inhomogeneous, with the same coefficient matrices in the left side as the system (3.3) and λ_ν as well as ξ_j, η_j, ζ_j ($j < \nu$) enter into the right sides of these systems. By satisfying the compatibility condition of these systems and expressing ξ_ν and η_ν in terms of ζ_ν therefrom, we obtain an equation for the variability f and equations for the intensity coefficients ζ_ν.

The equation for f is

$$f'^8 - \lambda_1 f'^4 + \frac{k^4}{R^2} = 0 \tag{3.4}$$

We hence obtain

$$f = \int_0^\beta \left(\frac{\lambda_1}{2} \pm \sqrt{D}\right)^{1/4} d\beta, \quad D = \frac{\lambda_1^2}{4} - \rho^4, \quad \rho = \frac{k}{\sqrt{R}} \tag{3.5}$$

First order linear differential equations are obtained for ζ_ν. The equation for ζ_1 is homogeneous and has the form

$$2\left(\frac{k^4}{f'^4 R} - Rf'^4\right)\frac{d\zeta_1}{d\beta} - \left(\frac{k^4 R'}{f'^4 R^2} + \frac{3k^4 f''}{f'^5 R} + 5Rf'^3 f''\right)\zeta_1 = 0 \tag{3.6}$$

The equations for ζ_ν ($\nu = 2, 3, \ldots$) are inhomogeneous with the same left sides as (3.6) but the λ_ν enter linearly into the right sides, as do functions of β already known from the calculation of ζ_ν. From these equations we obtain for ζ_ν

$$\zeta_1 = c_1 (f')^{1/2} D^{-1/4}, \quad \zeta_\nu = c_\nu (f')^{1/2} D^{-1/4} + \lambda_\nu F_\nu(\beta) + \Phi_\nu(\beta) \tag{3.7}$$

Here F_ν and Φ_ν are functions already known from the calculation of ζ_ν and c_ν are arbitrary constants. The equation for f' is of eighth order, hence, we obtain eight linearly-independent solutions (3.1) for each of the eight values of f, where λ_ν are assumed real and identical for each of the solutions. It follows from (3.5) that depending on the sign of D the function f can take on real, pure imaginary, and complex values corresponding to which the properties of the solutions (3.1) vary.

Taking a real combination of the eight solutions, let us proceed to solve the boundary value problem. For definiteness, let us set a condition of rigid support on the edges $\beta = 0$ and $\beta = \beta_0$

$$\xi = \eta = \zeta = d\zeta/d\beta = 0 \tag{3.8}$$

although the discussion will also be valid for other boundary conditions. The process of finding the λ_ν and the arbitrary constants $c_{\nu j}$ (j denotes the number of the solution and varies from one to eight) depends essentially on the sign of D.

Let us assume that $D > 0$ on $[0, \beta_0]$. Then f from (3.5) has four real and two pairs of pure imaginary values on $[0, \beta_0]$. Substituting the real combination of eight solutions into condition (3.8) and equating the coefficients for identical powers of ε in the eight series obtained, we obtain a set of systems of eight linear algebraic equations for $c_{\nu j}$ with parameters λ_ν. The system for c_{1j} is homogeneous, and equating the determinant to zero, we obtain a transcendental equation for λ_1 and we select its solution λ_1 which is of the order of unity, where $D > 0$ on $[0, \beta_0]$. The left sides of the algebraic systems for $c_{\nu j}$ ($\nu > 1$) are the same as for c_{1j}, but the λ_ν enter into the right sides, hence, by satisfying the compatibility condition for these systems we find λ_ν uniquely, and then by solving these systems we find the arbitrary constants.

In connection with the fact that (3.4) has two pairs of pure imaginary roots f', two positive, and two negative roots under the condition $D > 0$ it follows from (3.1), (3.5) and (3.7) that the vibrations modes are a linear combination of "distorted" sinusoids. Modes for the quarter and one-fifth frequencies correspond to these vibrations from the box shell vibrations modes pictured in Figs. 3 and 4.

Let $D < 0$ on $[0, \beta_0]$. Then it follows from (3.5) that all eight values of f are completely complex. If, as in the case for $D > 0$, an attempt is made to construct the process of finding $c_{\nu j}$ and λ_ν for the conditions (3.8), then it is easy to see that the equation for λ_1 has just a single real root, zero. Therefore, under the condition $D < 0$ on $[0, \beta_0]$ the lowest natural vibrations frequency has still not been achieved. The same deduction can be made if the conditions (3.8) are replaced by simple support conditions at $\beta = 0$ and $\beta = \beta_0$. (We note that a computation for a box shell has shown that if the roots of the characteristic equations on both EF and FG are all completely complex, then the shell does not vibrate).

Let us examine the case when λ_1 is such that D changes sign in $[0, \beta_0]$. For definiteness, let $D > 0$ in $[0, \beta_*)$, $D = 0$ for $\beta = \beta_*$ and $D < 0$ in $(\beta_*, \beta_0]$. At the point β_* it follows from the formula for f that (3.4) has four pairs of multiple roots

$$f'_{1,2} = i\left(\frac{\lambda_1}{2}\right)^{1/4}, \; f'_{3,4} = -i\left(\frac{\lambda_1}{2}\right)^{1/4}, \; f'_{5,6} = \left(\frac{\lambda_1}{2}\right)^{1/4}, \; f'_{7,8} = -\left(\frac{\lambda_1}{2}\right)^{1/4} \; (i^2 = -1)$$

The point at which (3.4) for f' has multiple roots is called a turning point. The behavior of solutions in the presence of a turning point has been studied in the monograph [3], for example; in connection with the problem of free vibrations of shells of revolution the turning points have been investigated (see [4] and others). The eight solutions of (3.1) − (3.7) are not linearly independent at the turning point. The solution in the neighborhood of this point must be constructed by the scheme given in the monograph [3], say, and must be merged with the solutions of (3.1) − (3.7), which are valid to the left and the right of this neighborhood. After the merger has been performed, the desired natural frequency can be found from the boundary conditions in a first approximation and the nature of the behavior of the vibrations modes in the neighborhood itself can be clarified. Knowing the nature of the values of f to the left and right of the neighborhood, the deduction can be made that the vibrations modes to the left of the turning point are "distorted" sinusoids, and to the right are oscillating, and damp out. It hence follows that if the turning point is not too close to β_0 then the solutions succeed in being damped sufficiently strongly upon approaching this edge, which means that the natural frequency in a first approximation does not depend on the boundary conditions at this edge. In the presence of a turning point the vibrations are concentrated on the section where primarily $D > 0$, and the section where $D < 0$ plays the part of a vibrations damper.

The vibrations in the presence of a turning point correspond to vibrations with the first frequency for the box shell vibrations modes ζ_* shown. The point F in Fig. 2 is not a turning point and only corresponds to one in the sense that the roots θ_1 on its left have two pairs of pure imaginary values, while all the roots θ_2 on its right are completely complex.

4. Let us examine the question of constructing the solutions of boundary value problems for quasi-transverse vibrations with medium variability. We turn to (3.4). If λ_1 is formally increased therein, then it separates into two equations. For relatively small roots, we obtain

$$-\lambda_1 f'^4 + \rho^4 = 0 \tag{4.1}$$

Hence it follows that

$$f_{1,2} = \pm\lambda_1^{-1/4}\int_0^\beta \rho\, d\beta, \qquad f_{3,4} = \pm i\lambda_1^{-1/4}\int_0^\beta \rho\, d\beta \tag{4.2}$$

For relatively large roots

$$f'^8 - \lambda_1 f'^4 = 0 \tag{4.3}$$

then

$$f_{1,2} = \pm \beta\lambda_1^{1/4}, \quad f_{3,4} = \pm i\beta\lambda_1^{1/4} \tag{4.4}$$

We arrive at (4.1) if the solution is sought in the form $(a, \ q/2$ are integers)

$$\begin{aligned}
\xi_* &= \exp(f / \varepsilon^a)\,(\varepsilon^{2a}\xi_1 + \varepsilon^{2a+1}\xi_2 + \ldots) \\
\eta_* &= \exp(f / \varepsilon^a)\,(\varepsilon^a\eta_1 + \varepsilon^{a+1}\eta_2 + \ldots.) \\
\zeta_* &= \exp(f / \varepsilon^a)\,(\zeta_1 + \varepsilon\zeta_2 + \ldots.) \\
\lambda &= \varepsilon^{4a}\lambda_1 + \varepsilon^{4a+1}\lambda_2 + \ldots, \ \varepsilon = (h / \sqrt{3(1-\sigma^2)})^{1/q}, \ a / q < 1/4
\end{aligned} \tag{4.5}$$

in place of (3.1), (3.2).

The process of finding the intensity coefficients is constructed as in Sect. 3. The expression for ζ_1 is hence the following (c_1 is an arbitrary constant):

$$\zeta_1 = c_1\rho^{1/2} \tag{4.6}$$

We obtain four linearly independent solutions (4.5) of the considered system of equations from (4.2) for each of the four values of f, but they are insufficient for compliance with the eight boundary conditions (we call these four solutions the fundamental integrals). Let us construct four more solutions which have the same asymptotic order λ as the fundamental integrals but a greater variability (we call these four solutions the auxiliary integrals corresponding to the fundamentals).

We seek the auxiliary integrals in the form

$$\begin{aligned}
\xi_* &= \exp(f / \varepsilon^b)\,(\varepsilon^{2b}\xi_1 + \varepsilon^{2b+1}\xi_2 + \ldots.) \\
\eta_* &= \exp(f / \varepsilon^b)\,(\varepsilon^b\eta_1 + \varepsilon^{b+1}\eta_2 + \ldots.) \\
\zeta_* &= \exp(f / \varepsilon^b)\,(\zeta_1 + \varepsilon\zeta_2 + \ldots.) \\
\lambda &= \varepsilon^{4a}\lambda_1 + \varepsilon^{4a+1}\lambda_2 + \ldots, \ \varepsilon = (h / \sqrt{3(1-\sigma^2)})^{1/q}, \quad b = (1/2)\,q - a
\end{aligned} \tag{4.7}$$

Formulas (4.7) result in equation (4.3) for f, which we discussed above. We obtain for ζ_1 from (4.7) that this quantity is an arbitrary constant, which means that taking (4.4) into account the conclusion can be reached that the auxiliary integrals are independent of the shell curvature in a first approximation while (4.2) and (4.6) show that the shell curvature affects the fundamental integrals in a first approximation.

It follows from (4.2) and (4.4) that there are two pairs of pure imaginary values (the remaining values are real) for the solution of the boundary value problems by using the fundamental and auxiliary integrals, just as in the case $D > 0$ on $[0, \beta_0]$ (see Sect. 3).

It has been shown in [5] for circular shells when $a = 0$ and $b / q = 1/2$, that there exist two kinds of vibrations. The vibrations of two kinds also hold for noncircular shells for λ from (4.5) upon compliance with the inequality from (4.5).

Let us show this. Denoting the real combination of fundamental integrals by P_1 and auxiliaries by P_2, we can satisfy the boundary conditions by using a solution P_3 which we define as follows:

$$P_3 = P_1 + \varepsilon^g P_2 \tag{4.8}$$

where g depends on the kind of boundary conditions and the kind of vibrations under consideration. Substitution of the solution (4.8) into the boundary conditions (3.8), after equating the coefficients of identical powers of ε in each of the equations obtained, yields an infinite set of linear algebraic systems for the arbitrary constants $c_{\nu j}$ ($\nu = 1, 2, \ldots, j = 1, 2, \ldots, 8$), in which the λ_ν enter as parameters. The system for c_{1j} is homogeneous. This system does not separate into subsystems in which the number of equations is greater or less than the number of unknowns for two values of the exponent: $g = 0$, $g = -b + a$. The eighth-order determinant for c_{1j} is the product of the fourth-order determinants.

In the case $g = 0$ the quantity λ_1 is found from the condition that the fourth-order determinant of the coefficients of those arbitrary constants which correpond to the fundamental integrals and the tangential boundary conditions $\xi = \eta = 0$, equals zero. The equation for λ_1 is

$$\cos(\lambda_1^{-1/4}\varepsilon^{-a}J) = 0 \qquad \left(J = \int_0^{\beta_0} \rho \, d\beta\right)$$

whence $\lambda_1 = ((N + 1/2)\pi\varepsilon^a / J)^{-4}$, where N is an integer of order ε^{-a}. The residuals obtained in the nontangential conditions $\zeta = d\zeta / d\beta = 0$ are eliminated by using the c_{1j} ($j = 5, \ldots, 8$) corresponding to the auxiliary integrals under the assumption that the determinant in these c_{1j} is not close to zero.

In the case $g = -b + a$ the quantity λ_1 is found upon compliance by the auxiliary integrals with the nontangential conditions from which we obtain the following equation for λ_1: $\cos(\lambda_1^{1/4}\varepsilon^{-b}\beta_0) = 0$, whence $\lambda_1 = ((T + 1/2)\pi\varepsilon^b / \beta_0)^4$, where T is an integer on the order of ε^{-b}. The residuals in the tangential boundary conditions are eliminated by using the fundamental integrals under the assumption that the determinant for c_{1j} ($j = 1, \ldots, 4$) is close to zero.

Following [5], let us call vibrations with $g = 0$ vibrations of type 1 and with $g = -b + a$ vibrations of type 2. We note that although the frequency for vibrations of type 1 in a first approximation is found from membrane problem, the predominant displacement ζ_* is substantially corrected over the whole range $[0, \beta_0]$ (due to the fact that the two values of f from (4.4) are pure imaginary) by the auxiliary integrals. For vibrations of type 2 both the first approximation of the frequency and the displacement ζ_* in a first approximation are well defined upon compliance with the nontangential boundary conditions by the auxiliary integrals. The succeeding λ_ν ($\nu > 1$), as well as $c_{\nu j}$ ($\nu > 1$), can also be found for vibrations of both types after compliance with the compatibility conditions and solving the system of equations for $c_{\nu j}$ ($\nu > 1$). There are no turning points and the associated phenomenon of primarily local vibrations for vibrations of both types.

We note that if a closed, smoothly curved shell is considered, then the interaction between the fundamental and auxiliary integrals violated and two independent modes of vibrations are obtained: one is constructed on the basis of solutions of the form (4.5) with pure imaginary values of f from (4.2), and the other on the basis of solutions of the form (4.7) with pure imaginary values of f from (4.4) (this can be shown by replacing the boundary conditions by eight periodicity conditions in the displacements and stress resultants).

It has been shown in [2] that both the quasi-transverse vibrations with high variability and the quasi-tangential vibrations depend slightly on the shell curvature, hence they will not be examined herein.

The method of exponential representation of the solution is not applicable to the solution of problems of noncircular cylindrical shell vibrations with low variability; such vibrations require special analysis.

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GENERALIZED CYCLIC DISPLACEMENTS

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We consider the generalized cyclic displacements of holonomic mechanical systems with a finite number of degrees of freedom, and their application to integration of the equations of motion.

N. G. Chetaev in [1] turned his attention to the formulation of problems dealing with general properties of mechanical systems and connected with the groups of transformations which leave the basic mechanical functions invariant. It was he who introduced [2] the concept of cyclic displacement of a mechanical system with smooth holonomic constraints. This concept was enlarged in [3] in the course of considering a particular case of motion of a mechanical system with three degrees of freedom.

1. Let us consider a mechanical system with smooth holonomic constraints, and with k degrees of freedom. We assume that the position of the system is determined by the real dependent variables $x_1, x_2, \ldots, x_n$ $(n > k)$. The possible displacements of this system are determined by an intransitive, k-membered group of infinitesimal operators

$$X_\alpha = \sum_{i=1}^{n} \xi_\alpha{}^i \frac{\partial}{\partial x_i} \qquad (\alpha = 1, 2, \ldots, k)$$

The problem of constructing the groups of possible displacements was studied in [4].